Creative Objectivism, a powerful alternative to Constructivism

Copyright © 2002 Paul P. Budnik Jr.
Mountain Math Software

Abstract

It is problematic to allow reasoning about infinite sets to be as unconstrained as that about finite sets. Yet Constructivism seems too restrictive in not allowing one to assume that an ideal computer program will either halt or not halt. Creative Objectivism considers as meaningful any property of integers which is determined by a recursively enumerable set of events. This captures the hyperarithmetical sets of integers and beyond thus including some sets that require quantification over the reals. This philosophy assumes mathematics is a human endeavor that creates objective truth by discovering meaningful properties of the integers that are determined by events that can be enumerated in a universe that may be potentially infinite. This philosophy leads to an "experimental" approach to extending mathematics.

1 Introduction

Mathematics strives for the most powerful and most general system possible. In the past this has led to inconsistent systems. Constructivists maintain that the only way to insure the correctness of mathematical reasoning about infinite sets is through constructive proofs. Proof by contradiction or the excluded middle is not allowed. This means that one cannot assume that an ideal computer program must either halt or not halt. Each step of an ideal computer program is logically determined. Assuming that the program will either halt in a finite time or run forever seems unproblematic. In general if a property of a computer program is determined by events enumerated in the execution of that program than the property would seem to be meaningful and objectively either true or false.

Central to this issue is the ontological status of mathematical abstractions. Finite sets can at least in theory exist as physical structures. In what sense are infinite sets meaningful? Properties of recursive processes or computer programs are objectively meaningful if they are determined by recursively enumerable finite events. All of the events could be physically realized in a potentially infinite universe.

This philosophical point of view does not see these properties as existing in some ideal state. They are created by human endeavor yet they are objectively determined. The recognition that these properties are objectively meaningful is a creative human act.

2 Truth

Ultimately truth is determined by physical consequences. We want to know what will happen if we make particular choices. A system that helps us to know this is to at least some degree true. One that misleads us is false. Abstract mathematics was developed because it helped us make predictions. But mathematical truth seems to be of a special absolute nature. $2 + 2 = 4$ is not so much a statement about the world as it is a statement about language. It follows from the definition of the terms. As long as mathematics is talking about finite objects it can be regarded as as stating what is true by definition.

Mathematical assertions that refer to all integers are useful. Each of the individual events such statements refer to is finite but the collection of all of them is infinite. Such assertions still seem to be true by definition. However we cannot be so certain about these statements. Their truth is dependent on laws of induction. There is little doubt that first order induction is valid. But we know from Gödel's Incompleteness Theorem that there will always be valid laws of induction that are outside of any formal mathematical system.

Logical truths about finite objects refer to something that can exist physically. Logical truths about all integers do not refer to any single object that could exist physically at least as far as we know. They are a creation of the human mind. However each of the events they refer to is a finite thing that can exist physically there is an objective basis for the truth of these created statements.

Existing mathematics in the form of Zermelo Frankel Set Theory provides powerful principles of induction by treating infinite sets in the same way it treats finite sets. Philosophically mathematicians have a variety of beliefs about the nature of these completed infinite totalities. Practically few would like to give up the enormous mathematical power that is implicit in this formalization of mathematics.

A problem with this approach to mathematics is the disconnect that exists between the higher axioms of mathematics and physical reality. It is our intuition about physical reality that is the starting point for mathematics. When the axioms become disconnected from reality how are we to develop our intuition about them?

The philosophical approach we are proposing keeps much of the power of contemporary mathematics intact and retains a connection with physical reality. Our definition of meaningful mathematical truth is imprecise. The problem is the phrase "determined by a recursively enumerable set of events." Deciding which statements meet that criteria is a creative act that ultimately will run up against the limitations of Gödel's Incompleteness Theorem. We discuss that in Section 7.

3 The hierarchy of mathematical truth

The Arithmetical and Analytical Hierarchies classify sets of integers based on the number of alternating quantifiers over the integers (arithmetical) and reals (analytical) in the statement that defines the set. For statements in the Arithmetical Hierarchy it is straight forward to enumerate all the finite events that determine if the statement is true or false. For a low level in the analytical hierarchy (Π^1_1) we can also do this. (A statement is Π^1_1 if it has a single pair of alternating quantifiers on the reals that begins with $\forall$.)

Let T_i be some Gödel numbering of TMs (Turing Machines) such that each TM accepts an integer input and after that may output an integer value. If a nonzero integer value is output it is interpreted as the Gödel number of a TM that has the same property of accepting an integer input and perhaps having a subsequent integer output. We define a path p as an arbitrary sequence of integers. $E(n, p, T_i)$ is the value output from T_i when evaluated on the first n elements of p. E may be undefined because T_i may terminate before we get to level n or because there is no output at that level.

Let P be the set of all infinite sequences of integers and ω be the set of all integers. We say that T_i is well founded iff the following holds.

$$\forall p \exists m \; (p \in P) \wedge (m \in \omega) \wedge E(m, p, T_i) = 0$$

This says that every path p terminates in some finite number m of steps.

The set of all well founded TMs is Π^1_1 complete[2, p 396]. This means from this set we can recursively compute any Π^1_1 set of integers.

We are able to enumerate all the events that determine if a TM is well founded. Those events are the $E(m, p.T_i)$ where p ranges over all integer sequences of length m and m ranges over all integers. Even though the set of well founded Turing Machines requires quantification over the reals to define it is an objective property of integers under Creative Objectivism.

4 Expanding Mathematics

We can continue to expand the hierarchy of meaningful properties of the integers but not by considering more alternating quantifiers on the reals. The natural extension is to consider TMs that are second order well founded. This means they are well founded not for infinite sequences of integers but for infinite sequences of first order well founded TMs as described above.

The property of being a well founded TM allows us to define and do induction on recursive ordinals. Recursive ordinals support the definition of powerful general forms of induction. No formal system can enumerate all recursive ordinals. However the property of being second order well founded allows us to do induction on *processes* for generating recursive ordinals. This can be extended in obvious and very subtle ways. We can use the structures we generate to provide notations to index more powerful forms of extending the definition. In standard set theory all these notations are ordinals. The concept of ordinal masks the complex combinatorial structure of what are recursive processes for generating other recursive processes of a given type. In limited ways this approach is a return to the explicit typology of *Principia Mathematica*. The big advantage we have today is the aid of computer technology to manage and experiment with the complex type hierarchy that is generated by this approach. We can construct the computer programs being typed and experiment with their properties.

The inductive strength of Creative Objectivism could eventually exceed that of ZF. This philosophy encourages computer models and experimentation as aids to developing mathematical intuition. That may have the potential to extend mathematics in ways that are beyond the reach of current approaches. I suspect that eventually the language of ZF will prove inadequate to describe all the mechanisms for iterating the idea that notations defined at one level of mathematics can be used to index the construction of new notations.

ZF is formlated as a language describing external entities. As Gödel proved one can model ZF wihtin ZF but constructing the model is a complicated exercize. A more natural approach with this philosophy is to have a language that explicitly recognizes that it is talking about itself. The notation schemes we define in the language are used to extend the language. Everything is a recursive process but the typology of the inputs and outputs of these processes vary and the language of the formal system defines that variation.

5 Time and creativity

Time is fundamental to our existence. Things change with time and there seems to be a creative element to time. The physical universe evolves over time. Biological evolution creates more complex creatures over time. Human culture advances in complexity over time.

The instinct in the mathematical search for absolute truth is to eliminate any fundamental philosophical significance to time. Sets exist absolutely. They do not exist in time. But if the world is as it seems this may be the wrong approach. Mathematics may not involve the discovery of preexisting infinite structures. It may instead involve the creation of objective truth about what may happen in a universe that could be potentially infinite but has no actual or completed infinite totalities. This suggests that mathematical truth can be both creative and experimental.

6 Mathematics excluded by Creative Objectivism

Although Creative Objectivism retains much of the power of contemporary mathematics in defining sets of integers it does not allow quantification over all sets. Some statements like the Continuum Hypothesis are not considered meaningful in any absolute sense. However the set of all reals definable in a particular formal system is a meaningful concept. The Continuum Hypothesis may be true false or undecidable within a formal system. Formal systems are perfectly valid objects of study in Creative Objectivism. After all they are TMs for enumerating theorems.

I doubt that any mathematics of practical value cannot be incorporated into Creative Objectivism. The continuum as the limit of finite processes is compatible. The logic of the calculus in developing derivatives and integrals does not present a problem. What would change is how the foundations of those disciplines are formulated.

7 Evolution and mathematics

This idea that mathematical truth can be both creative and objective suggests a new way of looking at the implications of Gödel's

Incompleteness Theorem. If mathematics is inherently creative it need not follow a single path of development. By following an ever increasing number of paths it is possible to develop mathematics in a way that is not subject to the limitations Gödel's result implies for single path processes. This is how the mathematically capable human mind was created. Biological evolution has followed an ever expanding number of paths without selecting a single "correct" or "true" path.

Divergent creative processes like biological evolution always involve tradeoffs between diversity and concentration of resources. By exploring the mathematics of creative divergent processes it may be possible to find boundary conditions for this tradeoff that optimize creativity. Such mathematics may be broadly applicable to many human endeavors.

In the long run mathematics will need to be extended through a creative divergent process if it is to avoid stagnation. There will be an ever expanding number of possible incompatible extensions to mathematics and schools will develop pursuing each of these with no expectation of any final resolution. The expectation will be for ever expanding diversity.

The wider implications of this philosophy are explored in the partial draft of a book availible online[1].

References

[1] Paul Budnik. *What is and what will be* WWW. Draft on web: www.mtnmath.com/willbe.html, Los Gatos, CA, 2002. 7

[2] Hartley Rogers Jr. *Theory of Recursive Functions and Effective Computability*. McGraw Hill, New York, 1967. 4