

Emergent properties of discretized wave equations

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Abstract

A computational TOE (theory of everything) requires a universe with discrete, as opposed to continuous, space, time and state. Any discrete model for the universe must approximate the continuous wave equation to extremely high accuracy. The wave equation plays a central role in physical theory. It is the solution to Maxwell's equations and the Klein Gordon (or Relativistic Schrödinger) Equation for a particle with zero rest mass. The interaction of waves as fields are a primary means of information transfer in the physical universe. Thus understanding the properties of discretized finite difference approximations to the wave equation is likely to play an important role in developing a digital TOE. In addition the equations are simple enough to be considered as a possible definition of a computational TOE.

No discrete equation can model the continuous wave equation exactly because the amplitude of a continuous wave decreases to an arbitrarily small value as it propagates. As a result discretization must introduce nonlinearities at something like the Planck time and distance scales. Thus detailed predictions about fundamental particles are difficult to make. However there are emergent properties that mimic aspects of quantum mechanics. One important result from these models is a possible explanation for apparent experimental violations of Bell's inequality. This explanation arises naturally from solving the problem of reconciling quantum randomness with absolute conservation laws.

1 Constraints on a digital theory of everything

A computational TOE (theory of everything) requires a universe with discrete, as opposed to continuous, space, time and state. Any discrete model for the universe must approximate the continuous wave equation to extremely high accuracy. The wave equation plays a central role in physical theory. It is the solution to Maxwell's equations and the Klein Gordon (or Relativistic Schrödinger) Equation for a particle with zero rest mass. The interaction of waves as fields are a primary means of information transfer in the physical universe. Thus understanding the properties of discretized finite difference approximations to the wave equation is likely to play an important role in developing a digital TOE. In addition the equations are simple enough to be considered as a possible definition of a computational TOE.

Discretized approximations to partial differential equations have been studied extensively but almost exclusively for the purpose of minimizing the difference between these models and the continuous equations they are designed to approximate. Here we are interested in how discretized finite difference approximations differ from the continuous wave equation.

2 Discrete approximations to the wave equation

The continuous wave equation with a propagation velocity of v and a solution of the form $f(x, y, z, t)$ is:

$$\frac{\partial^2 f}{\partial t^2} = v^2 \nabla^2 f. \quad (1)$$

The simplest finite difference approximation to the wave equation on a regular grid with three spatial dimensions and one time dimension defines a scalar value, $f_{i,j,k,t}$, at each point in space-time. With units chosen so that $\Delta x = \Delta t = 1$ the finite difference approximation is:

$$d = v^2(f_{i+1,j,k,t} + f_{i-1,j,k,t} + f_{i,j+1,k,t} + f_{i,j-1,k,t} + f_{i,j,k+1,t} + f_{i,j,k-1,t} - 6f_{i,j,k,t}) \quad (2)$$

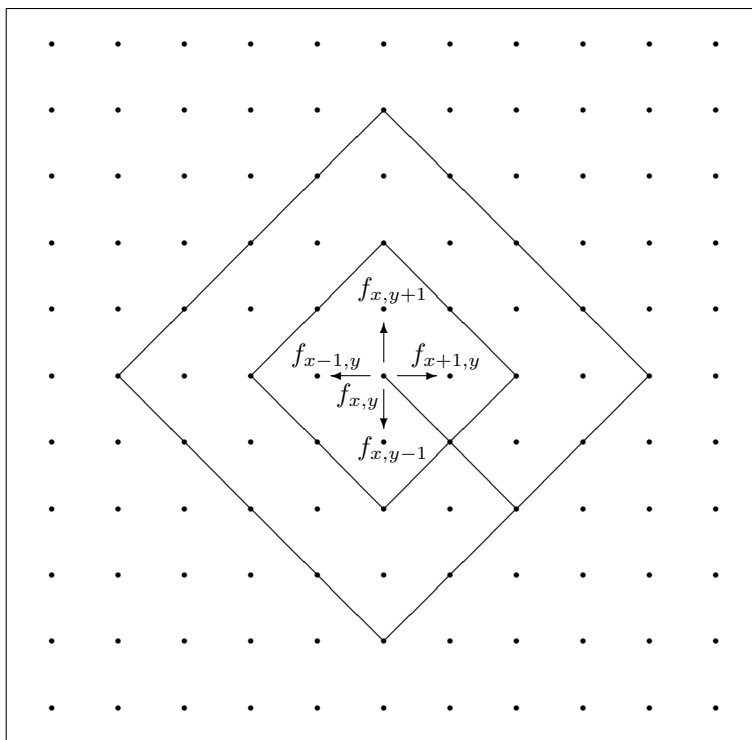
$$f_{i,j,k,t+1} = d - f_{i,j,k,t-1} + 2f_{i,j,k,t} \quad (3)$$

$f_{i,j,k,t+1}$ in Equation 3 is the value computed for the next time step. d in Equation 2 is the finite difference approximation to the spatial divergence (approximation to the right hand side of Equation 1). v is the velocity of propagation in units of the grid spacing in time and distance, i. e. $v = 1$ means the wave propagates one grid distance space in each time increment.

2.1 Approximating the isotropic continuous equation

To approximate the continuous wave equation, a wave front must advance in all directions at very close to the same speed. In the rectangular grid implied by Equations 2 and 3, the maximum rate at which effects can propagate is one immediate neighbor in one time step. The four arrows at the center of Figure 1 illustrate this. This figure also shows the diamond shaped propagation pattern of the first affected points at 2 and 4 time steps after a change in the center point.

A discrete rectangular grid cannot be isotropic and this puts constraints on the velocity of propagation in approximating the isotropic continuous case. The diagonal in Figure 1 from the center of the figure to the four time step propagation boundary has length $\sqrt{4+4}$ distance increments. Each side of the triangle with this line as hypotenuse is 2 distance increments. The distance to the same propagation boundary along a major axis is 4. The ratio is $\sqrt{8}/4 = 1/\sqrt{2}$. Assuming the standard Euclidean metric, effects can potentially propagate along a major axis by a factor of $\sqrt{2}$ faster than they can propagate at a 45 deg angle. Other grid topologies and finite difference



In equations 2 and 3 each point at each time step is affected only by immediate neighboring points as shown by the four arrows from the center point. The two diamonds are the boundaries of affected points after 2 and 4 time steps.

Figure 1: Maximum propagation of a physical effect

approximations will have different patterns and ratios, but no regular grid can have the complete circular symmetry of the continuous case and thus will require $v < 1$ to approximate the continuous equation.

2.2 Discretizing the finite difference equation

A discrete version of equations 2 and 3 must map integers to integers. Because $v < 1$, the equations must be modified to discretize them. The simplest thing is to truncate or round the computation to an integer value. Such a modification to Equation 2 will preserve the time symmetry of equations 3 and 1 and thus will be time reversible.

A truncated or rounded finite difference approximation to the wave equation is an inelegant model. There may be a more elegant solution to this problem, but any fully discrete TOE must have within it a discretized mechanism that approximates the wave equation. Thus it is worth understanding the emergent properties in this

class of models.

3 Emergent properties

The existing laws of physics break down near the Planck time and distance. At that scale background quantum fluctuations can create tiny black holes. This suggests that the minimal time and distance in a discrete space/time model will be close to the Planck time and distance. The Planck time is 5.4×10^{-44} seconds and the Planck distance is 1.6×10^{-35} meters. Discreteness at this scale would make for huge structures for the fundamental particles of physics. Thus, although it is easy to compute discrete approximations to the wave equation, they will inevitably, given existing technology, be much smaller than is required to reproduce known physics. However it is possible to develop general properties that characterize these models and to connect these to known physics.

Two classes of emergent properties that can arise from a discretized finite difference approximations to the wave equation are quantization (a lower limit on the size of viable structures which is shared by all discrete models) and chaotic like behavior from the nonlinearities of discretization.

3.1 Quantization

An initial nonzero state cannot diffuse to arbitrarily small values as is the case with the continuous equation. Because of time reversibility, an initial nonzero state cannot disappear. There are three options. The initial state can diverge, eventually producing nonzero values everywhere. It can break up into disjoint structures moving apart. Finally it can remain in place generating a sequence of states. For small initial states this last case will repeat a finite sequence of states that must include the initial state because of time reversibility. This is true of any state sequence that does not diverge or break up into independent structures, However, the number of possible states grows so rapidly with the size of the initial state, that one may never observe this looping.

3.2 Chaotic like behavior

Chaos theory's definition requires that different initial conditions be able to be specified arbitrarily close. This is only possible with continuous models, but, because discrete models can approximate continuous ones, there are chaotic-like discrete models that approximate chaotic behavior. The wave equation is linear and not chaotic. Discretizing it adds a nonlinear element to the model opening the possibility of chaotic like behavior.

Discretized approximations to the wave equation have solutions that are vastly more complex than the solutions to the continuous equation. This is easy to show for the scalar wave equation that is a function only of time. The solution to the

continuous equation is a sine wave completely determined by its phase, amplitude and frequency. The solution to a discretized version of this equation is a vast array of repeating sequences. Choose any two integers as the first two values of a sequence and they determine a sequence that will repeat sooner or later. The appendix gives some examples.

3.3 Particles

At this point this paper becomes speculative. I do not know what happens in these models on the scale of fundamental particles, because it is not possible to do simulations at that scale. Thus I describe what may happen based on properties of the models and existing physics. Such speculation may be essential to bridge the enormous gap between the toy discrete models that can be constructed with existing technology and the vast structures that correspond to fundamental particles if time and space are ultimately discrete.

On a large enough scale this class of models can mimic a solution to the partial differential equation to an arbitrarily high accuracy. It will have other characteristics that may be chaotic like. Specifically there may exist, in contrast to the continuous model, stable dynamic structures that are similar to attractors in chaos theory. They are structures that an initial state tends to converge to. Such structures could be the fundamental particles of physics. Under appropriate circumstances these structures could transform into one another creating particle interactions and transformations. This would be a physical quantum collapse that nonetheless is time reversible and thus never destroys information. It is possible that an appropriate discrete model could account for the entire menagerie of “fundamental particles” of contemporary physics as well as all the forces of nature. There is enough complexity introduced by nonlinear discretization to suggest that this is a possibility.

3.3.1 Special Relativity

All linear processes in this model can be thought of as electrodynamic. This implies that special relativity holds for all linear processes to the same accuracy that the model approximates the continuous wave equation. In contrast the transformation of particles is chaotic like, nonlinear and does not necessarily conform to special relativity.

3.3.2 Quantum uncertainty

Quantum uncertainty stems from the same mathematics that does not allow one to specify an exact frequency and exact location for a classical wave. Only an impulse has an exact location and it is the sum of all frequencies. Only a pure tone has an exact frequency and it has infinite extent. There is nothing ‘uncertain’ in classical wave theory. The class of discrete models proposed here suggests the possibility

that quantum uncertainty has a similar but more complex explanation that does not involve absolute uncertainty but only chaotic like behavior. The approach described below is for a discretized (and thus nonlinear and potentially chaotic like) finite difference approximation to the wave equation.

The idea is that the transformation of particles is a physical process extended over time in which dynamically stable structures (particles) transforms to other particles. Particles are like attractors in Chaos theory. They are the residual structures that processes ultimately resolve themselves into. These transformations have a focal point in physical space and state space. The sharpness of this focal point is determined by initial conditions and this is the basis of the uncertainty principle of quantum mechanics. As long as one only knows the properties of the system at the level of quantum mechanics, these focal points determine everything one can say about the system. However, with complete knowledge of the detailed discrete state, one could predict exactly what happens. At this lowest level of physical reality the uncertainty principle does not impose theoretical limitations on what one can know about a system, although the complexity and chaotic like structure of these transformations can impose enormous practical limitations.

3.3.3 Pseudo random behavior

This class of models is completely deterministic. However, the chaotic like behavior introduced through nonlinear discretization can create pseudo random results that in most cases would be impossible to distinguish from the irreducible randomness postulated by quantum mechanics. The “hidden variables” implied by Einstein and his coauthors critique of quantum mechanics[4] are not variables in the classical sense. They are information stored holographic like throughout an extended spatial region. They are the detailed evolution of a discretized time reversible finite difference approximation to the wave equation. They do not allow the assignment of exact values to non-commuting observables, but they do allow (at least in theory) prediction of all experimental outcomes.

4 Bell’s Inequality

The most problematic aspect of quantum mechanics is the reconciliation of absolute conservation laws with probabilistic observations. Bell proved that quantum mechanics requires that two particles that interact and remain in a singlet state are subject to mutual influence even if they are at opposite ends of the galaxy[1]. This influence is essential to insure that the conservation laws will hold.

One solution to this problem is an explicitly nonlocal model such as that developed by Bohm[2]. A discrete model can be nonlocal. This is a function of the way space time points are topologically connected[8, p. 544]. This solution adds complexity to the topological network that must grow with the size of the universe being modeled.

In addition it fails to address the underlying problem of probabilistic observations and absolute conservation laws. In quantum mechanics the mechanism that enforces the conservation laws is the evolution of the wave function in configuration space. This is an even more topologically complex model. It requires a *separate set of spatial dimensions for every particle* being modeled.

A chaotic like time reversible finite difference approximation to the wave equation may provide a simple local (but not relativistic) solution to the long standing paradox of probabilistic laws of observation and absolute conservation laws.

4.1 Creating the Universe

Moving beyond the scalar wave function to one or more spatial dimensions results in a huge number of possible states. One can no longer observe a repeated sequence of states in a toy model except for small initial states that spreads out little or not at all. What small means depends on the velocity of propagation, v , in Equation 2. If a single initial value, in an otherwise all zero grid, is large enough to propagate more than two distance intervals before looping through a repeated sequence of states, it usually diverges. The smaller the value of v the larger the initial value can be and not diverge. Divergence in this context means filling (a necessarily small)¹ state space with nonzero values. When I first observed the tendency of these simulations to diverge for all but very trivial initial conditions, I was concerned. Now I am less certain that is a problem. Like the old steady state model of the universe, it may be that matter is continually being created at an extremely small rate and the divergence I observed is that process. Divergent models allow for a small finite region of nonzero initial conditions and simple state change rules to account for the evolution of the universe. Everything is created from a very simple initial state as structure emerges from a divergent process.

4.2 A local mechanism for the conservation laws

The conservation laws, that characterize the continuous equation will, to high accuracy, be obeyed by the discretized model. These will impose constraints on what particle transformations can occur. One can expect many transformations to start to occur, but not be able to complete because of the conservation laws or other structural constraints. These could be a source of the virtual particles of quantum mechanics. The process of particle transformation is one of chaotic like convergence to a stable state as one set of attractor like structures (particles) converge to a new set of attractor like structures.

All transformations in this model are local but not necessarily relativistic. There is an absolute frame of reference (the discrete points in physical space) and there is a possibility of superluminal transfer of information. Define w as the velocity of one

¹A cube 1000 points in each dimension requires a billion sample array.

spatial point in one time step. It is the maximum rate at which effects can propagate regardless of the velocity of wave propagation that simulates the continuous equation. If c is the velocity of wave propagation in the same units as w , then w/c must be greater than 1. It can be much larger and may need to be for a sufficiently accurate approximation to the continuous equation. Thus the chaotic like transformations between particles, which do not exist in the continuous model, may involve the local but superluminal transfer of information.

4.3 Existing tests of Bell's Inequality

It is plausible that this class of models could account for the existing experimental tests of Bell's inequality. Early in this series of experiments it was observed that quantum mechanics has a property called delayed determinism. The outcome of an observation may not be determined until some time after the event occurred[5]. We see this explicitly in the creation of virtual particles. They come into existence before it is determined whether they will be allowed to exist. Quantum mechanics does not have an *objective* definition of event. That is why there are competing philosophical interpretations to fill this gap. Delayed determinism is an inherent part of the process of converging to a stable state. There is no absolute definition of stable. It is a matter of probabilities. Any transformation might ultimately be reversed although that probability may become negligibly small rapidly. Delayed determinism and the lack of an objective definition of event makes it tricky to say just when an experiment has conclusively violated Bell's Inequality.

There are significant weaknesses in most if not all existing experiments[6]. The process of converging to a stable state is a bit like water seeking its level. All the available mechanisms (paths of water flow) are used in creating the final state. The process of converging to a stable state may exploit every available loophole as well as superluminal effects possible with this class of models. If some finite difference approximation to the wave equation is a correct theory of everything, then eventually quantum mechanics will disagree with experimental results in tests of Bell's inequality. One can think of these experiments as trying to determine if there is a physical quantum collapse process with a local (but not relativistic) space-time structure that can be observed experimentally.

A recent test of Bell's Inequality was motivated by suggestions that there must be motion of a macroscopic object (with gravitational effects) for a measurement to be complete[7]. This experiment involved a spatial separation of the two detectors of 18 kilometers (or 60 microseconds). This set a new record for these experiments. Each detection triggered a voltage spike to a piezoelectric crystal that generated macroscopic mass movement in about 7.1 microseconds. The quantum efficiency was 10% and the dead time was 10 microseconds. This is not claimed to be a loophole free experiment, but it does illustrate the challenge that the models proposed here face.

5 Conclusions

Any computational model for physics must approximate the wave equation to high accuracy and no computational model can do so exactly. Discretizing this continuous equation creates nonlinearities that may produce chaotic like behavior and superluminal information transfer. The former may lead to stable dynamic structures that are a model for fundamental particles. These particle may transform into one another in a physical quantum collapse somewhat like transformations between attractors in chaos theory. The superluminal transfer of information in these models may be one component in existing experiments that appear to violate Bell's inequality in a way that precludes a local explanation.

Connecting this mathematics and speculation about it to physics is extremely difficult and likely to remain so for some time because of the vast computational resources that would be required to model the fundamental particles. This speculation suggests that one could look at tests of Bell's Inequality as tests for a physical quantum collapse process with a space-time structure. Evidence for such a structure could spur interest in digital physics.

References

- [1] J. S. Bell. On the Einstein Podolsky Rosen Paradox. *Physics*, 1:195, 1964. 6
- [2] David Bohm. A Suggested Interpretation of the Quantum Theory in terms of “Hidden” Variables, I and II. *Physical Review*, 85:166, 1952. 6
- [3] Paul Budnik. *What is and what will be: Integrating spirituality and science*. Mountain Math Software, Los Gatos, CA, 2006. 11
- [4] Albert Einstein, B. Podolsky, and N. Rosen. Can Quantum-mechanical Descriptions of Physical Reality be Considered Complete? *Physical Review*, 47:777, 1935. 6
- [5] James D. Franson. Bell’s Theorem and Delayed Determinism. *Physical Review D*, 31:2529, 1985. 8
- [6] P. G. Kwiat, P. H. Eberhard, A. M. Steinberg, and R. Y. Chiao. Proposal for a Loophole-free Bell Inequality Experiment. *Physical Review A*, 49:3209, 1994. 8
- [7] D. Salart, A. Baas, J. A. W. van Houwelingen, N. Gisin, and H. Zbinden. Spacelike Separation in a Bell Test Assuming Gravitationally Induced Collapses. *Physical Review Letters*, 100(22):220404, 2008. 8
- [8] Stephen Wolfram. *A New Kind of Science*. Wolfram Media, Inc, Champaign, IL, 2002. 6

Appendix

Discretized scalar wave equation²

The simplest dynamic discrete system involves a single scalar value that changes at each time step or iteration. A simple time symmetric finite difference equation for this is:

$$f_{t+1} = T(nf_t/d) - f_{t-1}. \quad (4)$$

n and d are integers (numerator and denominator). T is truncation toward 0.

The corresponding differential equation is:

$$\frac{d^2 f}{dt^2} = (n/d - 2)f(t). \quad (5)$$

The -2 comes in because the second order difference equation subtracts $(f_t - f_{t-1})$ from $(f_{t+1} - f_t)$ generating a term $-2f_t$.

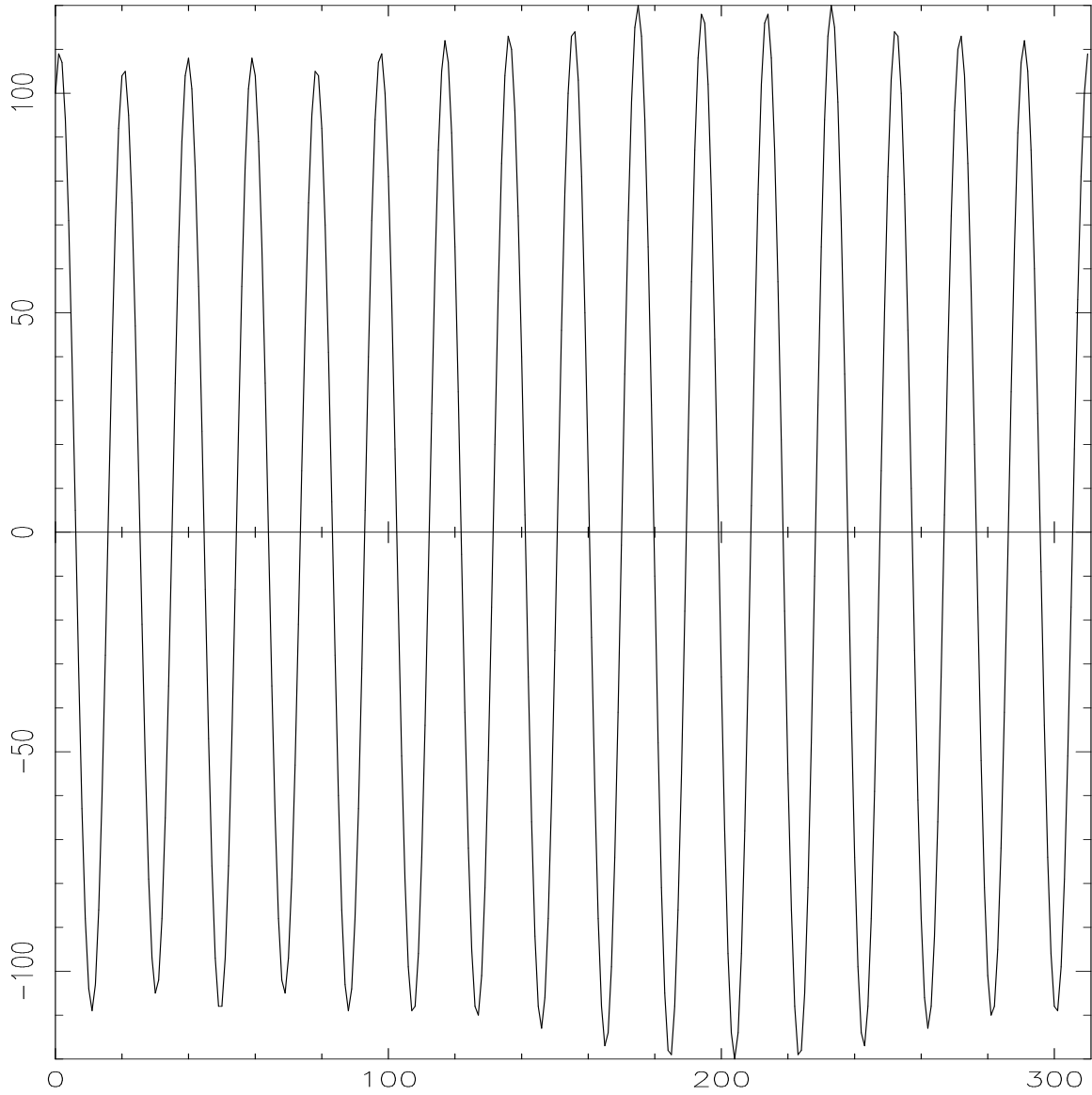
For $-2 < n/d < 2$ Equation 5 has a solution:

$$f(t) = \cos(t\sqrt{2 - n/d}). \quad (6)$$

t is in radians. Figure 2 plots a solution of the discretized finite difference equation for $n = 19$ and $d = 10$. This differs slightly from solution of the difference equation. The solution increments the angle of the \cos .3162 radians or 18.11° each time step. It takes about 20 time steps to complete one cycle of the \cos wave.

The solutions to the finite difference equation (4) are more complex than the solutions to the corresponding differential equation (5). The latter are completely described by equations like 6. The former have a rich structure that varies with the initial conditions. Table 1 gives the length until the sequence starts to repeat the solution to Equation 4 (again with $n = 19$ and $d = 10$) for various initial conditions.

²Excerpted from [3]



The above is a plot of the solution to $f_{t+1} = T(nf_t/d) - f_{t-1}$ with $f_0 = 100$, $f_1 = 109$, $n = 19$ and $d = 10$. T is truncation toward 0. It completes about 5 cycles for every 100 iterations. It departs significantly from the solution to the differential equation.

Figure 2: Simple discretized finite difference equation plot

	f_1												
f_0	100	101	102	103	104	105	106	107	108	109	110	111	112
100	154	269	154	154	269	328	328	328	289	309	174	116	116
101	269	77	250	328	250	289	328	77	309	116	309	174	174
102	154	250	154	289	328	309	250	77	77	58	174	289	77
103	154	328	289	77	328	328	328	328	77	309	116	289	289
104	269	250	328	328	77	309	289	250	309	309	289	174	174
105	328	289	309	328	309	328	77	77	289	58	289	77	309
106	328	328	250	328	289	77	116	77	77	116	116	174	77
107	328	77	77	328	250	77	77	58	289	309	174	289	309
108	289	309	77	77	309	289	77	289	309	309	309	289	174
109	309	116	58	309	309	58	116	309	309	289	174	174	289
110	174	309	174	116	289	289	116	174	309	174	250	116	77
111	116	174	289	289	174	77	174	289	289	174	116	174	174
112	116	174	77	289	174	309	77	309	174	289	77	174	116
113	309	135	289	251	309	116	309	174	309	116	309	251	289
114	58	174	58	309	135	135	174	289	77	289	174	135	135
115	174	174	251	174	174	174	251	116	174	58	174	116	251
116	58	174	309	135	77	251	58	77	58	135	174	135	58
117	251	174	251	251	251	58	135	174	309	135	251	174	251
118	58	251	406	58	135	309	251	251	309	174	58	77	135
119	484	368	174	58	58	174	174	58	309	251	174	251	251
120	368	232	484	406	368	368	58	251	58	174	135	251	58
121	58	232	232	406	406	484	58	174	97	251	174	58	174
122	58	58	484	232	484	58	406	368	406	58	174	58	58
123	406	194	484	232	484	368	58	484	484	368	58	406	58
124	213	155	174	368	232	232	58	97	484	368	58	58	368
125	194	155	174	406	97	213	484	58	368	232	484	406	58
126	349	136	213	174	174	97	232	368	58	97	484	232	232

The above gives the length until repetition of the sequence generated by $f_{t+1} = T(nf_t/d) - f_{t-1}$ with $n = 19$ and $d = 10$. T is truncation toward 0. The table is symmetric about a diagonal because reversing the order of the initial two values does not change the sequence length. It reverses the sequence order because the equation is symmetric in time.

Table 1: Cycle lengths for discretized finite difference equation