Formalizing objective mathematics

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1 Introduction

Objective mathematics attempts to distinguishes between statements that are objectively true or false and those that are only true, false or undecidable relative to a particular formal system. This distinction is based on the assumption of an always finite but perhaps potentially infinite universe[1]. This is a return to the earlier conception of mathematical infinity as a potential that can never be realized. This is not to ignore the importance and value of the algebra of the infinite that has grown out of Cantor's approach to mathematical infinity. It does suggest a reinterpretation of those results in terms of the countable models implied by the Löwenheim-Skolem theorem. It also suggests approaches for expanding the foundations of mathematics that include using the computer as a fundamental research tool. These approaches may be more successful at gaining wide spread acceptance than large cardinal axioms which are far removed from anything physically realizable.

1.1 The mathematics of recursive processes

A core idea is that only the mathematics of finite structures and properties of recursive processes is objective. This does not include uncountable sets, but it does include much of mathematics including some statements that require quantification over the reals[4]. For example, the question of whether a recursive process defines a notation for a recursive ordinal requires quantification over the reals to state but is objective.

Loosely speaking objective properties of recursive processes are those logically determined by a recursively enumerable sequence of events. This cannot be precisely formulated, but one can precisely state which set definitions in a formal system meet this criteria (see Section 5).

The idea of objective mathematics is closely connected to generalized recursion theory. The latter starts with recursive relations and expands these with quantifiers over the integers and reals. As long as the relations between the quantified variable are recursive, the events that logically determine the result are recursively enumerable.

1.2 The uncountable

It is with the uncountable that contemporary set theory becomes incompatible with infinity as a potential that can never be realized. Proving something is true for all entities that meet some property does not require that a collection of all objects that satisfy that property exists. Real numbers exist as potentially infinite sequences that are either recursively enumerable or defined by a non computable, but still logically determined, mathematical expression. The idea of the collection of all reals is closely connected with Cantor's proof that the reals are not countable. For that proof to work reals must exist as completed infinite decimal

[1] A potentially infinite universe is one containing a finite amount of information at any point in time but with unbounded growth over time in its information content.

expansions or some logically equivalent structure. This requires infinity as an actuality and not just a potential.

This paper is a first attempt to formally define which statements in Zermelo Frankel set theory (ZF)[6] are objective. The goal is not to offer a weaker alternative. ZF has an objective interpretation in which all objective questions it decides are correctly decided. The purpose is to offer a new interpretation of the theory that seems more consistent with physical reality as we know it. This interpretation is relevant to extending mathematics. Objective questions have a truth value independent of any formal system. If they are undecidable in existing axiom systems, one might search for new axioms to decide them. In contrast there is no basis on which relative questions, like the continuum hypothesis, can be objectively decided.

1.3 Expanding the foundations of mathematics

Defining objective mathematics may help to shift the focus for expanding mathematics away from large cardinal axioms. Perhaps in part because they are not objective, it has not been possible to reach consensus about using these to expand mathematics. An objective alternative is to expand the hierarchy of recursive and countable ordinals by using computers to deal with the combinatorial explosion that results[5].

Throughout the history of mathematics, the nature of the infinite has been a point of contention. There have been other attempts to make related distinctions. Most notable is intuitionism stated by Brouwer[1]. These approaches can involve (and intuitionism does involve) weaker formal systems that allow fewer questions to be decidable and with more difficult proofs. Mathematicians consider Brouwer's approach interesting and even important but few want to be constrained by its limitations. A long term aim of the approach of this paper is to define a formal system that is widely accepted and is stronger than ZF in deciding objective mathematics.

2 Background

The most widely used formalization of mathematics, Zermelo Frankel set theory plus the axiom of choice (ZFC)[6], gives the same existential status to every object from the empty set to large cardinals. Finite objects and structures can exist physically. As far as we know this is not true of any infinite objects. Our universe could be potentially infinite but it does not seem to harbor actual infinities.

This disconnect between physical reality and mathematics has long been a point of contention. One major reason it has not been resolved is the power of the existing mathematical framework to solve problems that are relevant to a finite but potentially infinite universe and that cannot be solved by weaker systems. A field of mathematics has been created, called reverse mathematics, to determine the weakest formal system that can solve specific problems. There are problems in objective mathematics that have been shown to be solvable only by using large cardinal axioms that extend ZF. This however has not resulted in widespread acceptance of such axioms. For one thing there are weaker axioms (in terms of definability) that could solve these problems. There do not exist formal systems limited to objective

mathematics that include such axioms[2] in part because of the combinatorial complexity they require. Large cardinal axioms are a simpler and more elegant way to accomplish the same result, but one can prove that alternatives exist. Mathematics can be expanded at many levels in the ordinal hierarchy. Determining the minimal ordinal that decides some question is very different from determining the minimum formal system that does so.

2.1 The ordinal hierarchy

Ordinal numbers generalize induction on the integers. As such they form the backbone of mathematics. Every integer is an ordinal. The set of all integers, ω, is the smallest infinite ordinal. There are three types of ordinals: 0 (or the empty set), successors and limits. ω is the first limit ordinal. The successor of an ordinal is defined to be the union of the ordinal with itself. Thus for any two ordinals a and b $a < b \equiv a \in b$. This is very convenient, but it masks the rich combinatorial structure required to define finite ordinal notations and the rules for manipulating them.

From an objective standpoint it is more useful to think of ordinals as properties of recursive processes. The recursive ordinals are those whose structure can be enumerated by a recursive process. For any recursive ordinal, R, on can define a unique sequence of finite symbols (a notation) to represent each ordinal $\leq R$. For these notations one can define a recursive process that evaluates the relative size of any two notations and a recursive process that enumerates the notations for all ordinals smaller than that represented by any notation.

Starting with the recursive ordinals there are many places where the hierarchy can be expanded. It appears that the higher up the ordinal hierarchy one works, the stronger the results that can be obtained for a given level of effort. However, I suspect, and history suggests, that the strongest results will ultimately be obtained by working out the details at the level of recursive and countable ordinals. These are the objective levels.

2.2 The *true* power set

Going beyond the countable ordinals with the power set axiom moves beyond objective mathematics. No formal system can capture what mathematicians want to mean by the *true* power set of the integers or any other uncountable set. This follows from Cantor's proof that the reals are not countable and the Löwenheim-Skolem theorem that established that every formal system that has a model must have a countable model. The collection of all the subsets of the integers provably definable in ZF is countable. Of course it is not countable *within* ZF. The union of all sets provably definable by any large cardinal axiom defined now or that ever can be defined in any possible finite formal system is countable. One way some mathematicians claim to get around this is to say the *true* ZF includes an axiom for every

[2]Axioms that assert the existence of large recursive ordinals can provide objective extensions to objective formalizations of mathematics. Large cardinal axioms imply the existence of large recursive ordinals that can solve many of the problems currently only solvable with large cardinals. However, deriving explicit formulations of the recursive ordinals provably definable in ZF alone is a task that has yet to be completed. With large cardinal axioms one implicitly defines larger recursive ordinals than those provably definable in ZF.

true real number asserting its existence. This is a bit like the legislator who wanted to pass a law that π is 3 1/7. You can make the law but you cannot enforce it.

My objections to ZF are not to the use of large cardinal axioms, but to some of the philosophical positions associated with them and the practical implications of those positions[3]. Instead of seeing formal systems for what they are, recursive processes for enumerating theorems, they are seen by some as as transcending the finite limits of physical existence. In the Platonic philosophy of mathematics, the human mind transcends the limitations of physical existence with direct insight into the nature of the infinite. The infinite is not a potential that can never be realized. It is a Platonic objective reality that the human mind, when properly trained, can have direct insight into.

This raises the status of the human mind and, most importantly, forces non mental tools that mathematicians might use into a secondary role. This was demonstrated when a computer was used to solve the long standing four color problem because of the large number of special cases that had to be considered. Instead of seeing this as a mathematical triumph that pointed the way to leveraging computer technology to aid mathematics, there were attempts to delegitimize this approach because it went beyond what was practical to do by human mental capacity alone.

Computer technology can help to deal with the combinatorial explosion that occurs in directly developing axioms for large recursive ordinals[5]. Spelling out the structure of these ordinals is likely to provide critical insight that allows much larger expansion of the ordinal hierarchy than is possible with the unaided human mind even with large cardinal axioms. If computers come to play a central role in expanding the foundations of mathematics, it will significantly alter practice and training in some parts of mathematics.

3 Mathematical Objects

In the philosophical framework of this paper there are three types of mathematical 'objects':

1. finite sets,
2. properties of finite sets and
3. properties of properties.

Finite sets are abstract idealizations of what can exist physically.

Objective properties of finite sets, like being an even integer, are logically determined human creations. Whether a particular finite set has the property follows logically from the definition of the property. However, the property can involve questions that are logically determined but not determinable. One example is the set of Gödel numbers of Turing Machines (TM)[3] that do not halt. What the TM does at each time step is logically determined and thus so is the question of whether it will halt, but, if it does not halt, there is no general

[3]There is overwhelming evidence that one can create a Universal Turing Machine that can simulate every possible computer. Assuming this is true, one can assign a unique integer to every possible program for this universal computer. This is called a Gödel number because Gödel invented this idea in a different, but related, context a few years before the concept of a Universal Turing Machine was proposed.

way to determine this. The non-halting property is objectively determined but not in general determinable.

3.1 Properties of properties

The property of being a subset of the integers has led to the idea that uncountable collecitons exist. No finite or countable formal system can capture what mathematicians want to mean by the set of all subsets of the integers. One can interpret this as the human mind transcending the finite or that mathematics is a human creation that can always be expanded. The inherent incompleteness of any sufficiently strong formal system similarly suggests that mathematics is creative.

3.2 Gödel and mathematical creativity

Gödel proved that any formalization of mathematics strong enough to embed the primitive recursive functions (or alternatively a Universal Turing Machine) must be either incomplete or inconsistent[7]. In particular such a system, will be able to model itself, and will not be able to decide if this model of itself is consistent unless it is inconsistent.

This is often seen as putting a limit on mathematical knowledge. It limits what we can be certain about in mathematics, but not what we can explore. A divergent creative process can, in theory, pursue every possible finite formalization of mathematics as long as it does not have to choose which approach or approaches are correct. Of course it can rule out those that are discovered to be inconsistent or to have other provable flaws. This may seem to be only of theoretical interest. However the mathematically capable human mind is the product of just such a divergent creative process known as biological evolution.

3.3 Cantor's incompleteness proof

One can think of Cantor's proof (when combined with the Löwenheim-Skolem theorem) as the first great incompleteness proof. He proved the properties defining real numbers can always be expanded. The standard claim is that Cantor proved that there are "*more*" reals than integers. That claim depends on each real existing as a completed infinite totality. From an objective standpoint, Cantor's proof shows that any formal system that meets certain prerequisites can be expanded by diagonalizing the real numbers provably definable within the system. Of course this can only be done from outside the system.

Just as one can always define more real numbers one can always create more objective mathematics. One wants to include as objective all statements logically determined by a recursively enumerable sequence of events, but that can only be precisely defined relative to a particular formal system and will always be incomplete and expandable just as the reals are.

4 Axioms of ZFC

The formalization of objectivity starts with the axioms of Zermelo Frankel Set Theory plus the axiom of choice ZFC, the most widely used formalization of mathematics. This is not the ideal starting point for formalizing objective mathematics but it is the best approach to clarify where in the existing mathematical hierarchy objective mathematics ends. To that end a restricted version of these axioms will be used to define an objective formalization of mathematics.

The following axioms are adapted from *Set Theory and the Continuum Hypothesis*[6][4].

4.1 Axiom of extensionality

Without the axiom that defines when two sets are identical (=) there would be little point in defining the integers or anything else. The axiom of extensionality says sets are uniquely defined by their members.

$$\forall x \forall y \;\; (\forall z \;\; z \in x \equiv z \in y) \equiv (x = y)$$

This axiom[5] says a pair of sets x and y are equal if and only if they have exactly the same members.

4.2 Axiom of the empty set

The empty set must be defined before any other set can be defined.

The axiom of the empty set is as follows.

$$\exists x \forall y \;\; \neg(y \in x)$$

This axiom[6] says there exists an object x that no other set belongs to. x contains nothing. The empty set is denoted by the symbol $\emptyset$.

4.3 Axiom of unordered pairs

From any two sets x and y one can construct a set that contains both x and y. The notation for that set is $\{x, y\}$.

$$\forall x \forall y \; \exists z \; \forall w \;\; w \in z \equiv (w = x \vee w = y)$$

This says for every pair of sets x and y there exists a set w that contains x and y and no other members. This is written as $w = x \cup y$.

[4]The axioms use the existential quantifier ($\exists$) and the universal quantifier ($\forall$). $\exists x \, g(x)$ means there exists some set x for which $g(x)$ is true. Here $g(x)$ is any expression that includes x. $\forall x \, g(x)$ means g(x) is true of every set x.

[5]$a \equiv b$ means a and b have the same truth value or are equivalent. They are either both true or both false. It is the same as $(a \rightarrow b) \wedge (b \rightarrow a)$.

[6]The '¬' symbol says what follow is not true.

4.4 Axiom of union

A set is an arbitrary collection of objects. The axiom of union allows one to combine the objects in many different sets and make them members of a single new set. It says one can go down two levels taking not the members of a set, but the members of members of a set and combine them into a new set.

$$\forall x \exists y \; \forall z \;\; z \in y \equiv (\exists t \; z \in t \wedge t \in x)$$

This says for every set x there exists a set y that is the union of all the members of x. Specifically, for every z that belongs to the union set y there must be some set t such that t belongs to x and z belongs to t.

4.5 Axiom of infinity

The integers are defined by an axiom that asserts the existence of a set ω that contains *all* the integers. ω is defined as the set containing 0 and having the property that if n is in ω then $n+1$ is in ω. From any set x one can construct a set containing x by constructing the unordered pair of x and x. This set is written as $\{x\}$.

$$\exists x \, \emptyset \in x \wedge [\forall y \, (y \in x) \rightarrow (y \cup \{y\} \in x)]$$

This says there exists a set x that contains the empty set $\emptyset$ and for every set y that belongs to x the set $y+1$ constructed as $y \cup \{y\}$ also belongs to x.

The axiom of infinity implies the principle of induction on the integers.

4.6 Axiom scheme of replacement

The axiom scheme for building up complex sets like the ordinals is called replacement. They are an easily generated recursively enumerable infinite sequence of axioms.

The axiom of replacement scheme describes how new sets can be defined from existing sets using any relationship $A_n(x, y)$ that defines y as a function of x. A function maps any element in its range (any input value) to a unique result or output value.

$\exists \, ! \, y \, g(y)$ means there exists one and only one set y such that $g(y)$ is true. The axiom of replacement scheme is as follows.

$$B(u, v) \equiv [\forall y (y \in v \equiv \exists x [x \in u \wedge A_n(x, y)])]$$

$$[\forall x \exists \, ! \, y A_n(x, y)] \rightarrow \forall u \exists v (B(u, v))$$

That first line defines $B(u, v)$ as equivalent to $y \in v$ if and only if there exists an $x \in u$ such that $A_n(x, y)$ is true. One can think of $A_n(x, y)$ as defining a function that may have multiple values for the same input. $B(u, v)$ says v is the image of u under this function.

This second line says if A_n defines y uniquely as a function of x then the for all u there exists v such that $B(u.v)$ is true.

This axioms says that, if $A_n(x, y)$ defines y uniquely as a function of x, then one can take any set u and construct a new set v by applying this function to every element of u and taking the union of the resulting sets.

This axiom schema came about because previous attempts to formalize mathematics were too general and led to contradictions like the Barber Paradox[7]. By restricting new sets to those obtained by applying well defined functions to the elements of existing sets it was felt that one could avoid such contradictions. Sets are explicitly built up from sets defined in safe axioms. Sets cannot be defined as the *universe* of all objects satisfying some relationship. One cannot construct the set of all sets which inevitably leads to a paradox.

4.7 Power set axiom

The power set axiom says the set of all subsets of any set exists. This is not needed for finite sets, but it is essential to define the set of all subsets of the integers.

$$\forall x \exists y \forall z [z \in y \equiv z \subseteq x]$$

This says for every set x there exists a set y that contains all the subsets of x. z is a subset of x ($z \subseteq x$) if every element of z is an element of x.

The axiom of the power set completes the axioms of ZF or Zermelo Frankel set theory. From the power set axiom one can conclude that the set of all subsets of the integers exists. From this set one can construct the real numbers.

This axioms is necessary for defining recursive ordinals which is part of objective mathematics. At the same time it allows for questions like the continuum hypothesis that are relative. Drawing the line between objective and relative properties is tricky.

4.8 Axiom of Choice

The Axiom of Choice is not part of ZF. It is however widely accepted and critical to some proofs. The combination of this axiom and the others in ZF is called ZFC.

The axiom states that for any collection of non empty sets C there exists a choice function f that can select an element from every member of C. In other words for every $e \in C$ $f(e) \in e$.

$$\forall C \exists f \forall e [(e \in C \wedge e \neq \emptyset) \rightarrow f(e) \in e]$$

4.9 The axioms of ZFC summary

1. Axiom of extensionality (See Section 4.1).

$$\forall x \forall y \ \ (\forall z \ \ z \in x \equiv z \in y) \equiv (x = y)$$

2. Axiom of the empty set (See Section 4.2).

$$\exists x \forall y \ \ \neg (y \in x)$$

[7] The barber paradox concerns a barber who shaves everyone in the town except those who shave themselves. If the barber shaves himself then he must be among the exceptions and cannot shave himself. Such a barber cannot exist.

3. Axiom of unordered pairs (See Section 4.3).

$$\forall x \forall y \; \exists z \; \forall w \;\; w \in z \equiv (w = x \vee w = y)$$

4. Axiom of union (See Section 4.4).

$$\forall x \exists y \; \forall z \;\; z \in y \equiv (\exists t \; z \in t \wedge t \in x)$$

5. Axiom of infinity (See Section 4.5).

$$\exists x \, \emptyset \in x \wedge [\forall y \, (y \in x) \rightarrow (y \cup \{y\} \in x)]$$

6. Axiom schema of replacement (See Section 4.6).

$$B(u, v) \equiv [\forall y (y \in v \equiv \exists x [x \in u \wedge A_n(x, y)])]$$

$$[\forall x \exists \, ! \, y A_n(x, y)] \rightarrow \forall u \exists v (B(u, v))$$

7. Axiom of the power set (See Section 4.7).

$$\forall x \exists y \forall z [z \in y \equiv z \subseteq x]$$

8. Axiom of choice (See Section 4.8).

$$\forall C \exists f \forall e [(e \in C \wedge e \neq \emptyset) \rightarrow f(e) \in e]$$

5 The Objective Parts of ZF

Objectivity is a a property of set definitions. Its domain is expressions within ZF (or any formalization of mathematics) that define new sets (or other mathematical objects). A set is said to be objective if it can be defined by an objective statement.

The axiom of the empty set and the axiom of infinity are objective. The axiom of unordered pairs and the axiom of union are objective when they define new sets using only objective sets. The power set axiom applied to an infinite set is not objective and it is unnecessary for finite sets.

A limited version of the axiom of replacement is objective. In this version the formulas that define functions (the A_n in this paper) are limited to recursive relations on the bound variables and objective constants. Both universal and existential quantifiers are limited to ranging over the integers or subsets of the integers. Without the power set axiom, the subsets of the integers do not form a set. However the property of being a subset of the integers

$$S(x) \equiv \forall_y y \in x \rightarrow y \in \omega$$

can be used to restrict a bound variable.

Quantifying over subsets of the integers suggests searching through an uncountable number of sets. However, by only allowing a recursive relation between bound variables and objective constants, one can enumerate all the events that determine the outcome. A computer program that implements a recursive relationship on a finite number of subsets of the integers must do a finite number of finite tests so the result can be produced in a finite time. A nondeterministic computer program[8] can enumerate all of these results. For example the formula

$$\forall r S(r) \rightarrow \exists n(n \in \omega \ \rightarrow a(r,n))$$

is determined by what a recursive process does for every *finite* initial segment of every subset of the integers[9]. One might think of this approach as a few steps removed from constructivism. One does not need to produce a constructive proof that a set exists. One does need to prove that every event that determines the members of the set is constructible.

6 Formalization of the Objective Parts of ZF

Following are axioms that define the objective parts of ZF as outlined in the previous section. The purpose is not to offer a weaker alternative to ZF but to distinguish the objective and relative parts of that system.

6.1 Axioms unchanged from ZF

As long as the universe of all sets is restricted to objective sets the following axioms are unchanged from ZF.

1. Axiom of extensionality

$$\forall x \forall y \ \ (\forall z \ \ z \in x \equiv z \in y) \equiv (x = y)$$

2. Axiom of the empty set

$$\exists x \forall y \ \ \neg(y \in x)$$

3. Axiom of unordered pairs

$$\forall x \forall y \ \exists z \ \forall w \ \ w \in z \equiv (w = x \vee w = y)$$

4. Axiom of union

$$\forall x \exists y \ \forall z \ \ z \in y \equiv (\exists t \ z \in t \wedge t \in x)$$

5. Axiom of infinity

$$\exists x \, \emptyset \in x \wedge [\forall y \, (y \in x) \rightarrow (y \cup \{y\} \in x)]$$

[8]In this context nondeterministic refers to a a computer that simulates many other computer programs by emulating each of them and switching in time between them in such a way that every program is fully executed. The emulation of a program stops only if the emulated program halts. The programs being emulated must be finite or recursively enumerable. In this context nondeterministic does not mean unpredictable.

[9]Initial segments of subsets of the integers are ordered and thus defined by the size of the integers.

6.2 Objective axiom of replacement

In the following A_n is any recursive relation in the language of ZF in which constants are objectively defined and quantifiers are restricted to range over the integers (ω) or be restricted to subsets of the integers. Aside from these restrictions on A_n, the objective active of replacement is the same as it is in ZF.

$$B(u,v) \equiv [\forall y(y \in v \equiv \exists x[x \in u \wedge A_n(x,y)])]$$
$$[\forall x \exists\, !\, y A_n(x,y)] \rightarrow \forall u \exists v(B(u,v))$$

7 An Objective Interpretation of ZFC

I suspect it is consistent to assume the power set axiom in ZF, because all subsets of the integers (and larger cardinals) that are *provably definable in ZF* form a definite, albeit countable, collection. These are definite collections only relative to a specific formal system. Expand ZF with an axiom like "there exists an inaccessible cardinal" and these collections expand.

Uncountable sets in ZF suggest how the objective parts of ZF can be expanded. Create an explicitly countable definition of the countable ordinals defined by the ordinals that are uncountable within ZF. Expand ZF to ZF+ with axioms that assert the existence of these structures. This approach to expansion can be repeated with ZF+. The procedure can be iterated and it must have a fixed point that is unreachable with these iterations.

Ordinal collapsing functions[2] do something like this. They use uncountable ordinals as notations for recursive ordinals to expand the recursive ordinals. Ordinal collapsing can also use countable ordinals larger than the recursive ordinals. This is possible at multiple places in the ordinal hierarchy. I suspect that uncountable ordinals provide a relatively weak way to expand the recursive and larger countable ordinal hierarchies. The countable ordinal hierarchy is a bit like the Mandelbrot set[8]. The hierarchy definable in any particular formal system can be embedded within itself at many places.

The objective interpretation of ZFC see it as a recursive process for defining finite sets, properties of finite sets and properties of properties. These exist either as physical objects that embody the structure of finite sets or as expressions in a formal language that can be connected to finite objects and/or expressions that define properties. Names of all the objects that provably satisfy the definition of any set in ZF are recursively enumerable because all proofs in any formal system are. These names and their relationships form an interpretation of ZF.

8 A Creative Philosophy of Mathematics

Platonic philosophy visualizes an ideal realm of absolute truth and beauty of which the physical world is a dim reflection. This ideal reality is perfect, complete and thus static. In stark contrast, the universe we inhabit is spectacularly creative. An almost amorphous cosmic big bang has evolved into an immense universe of galaxies each of which is of a size and complexity that takes ones breath away. On at least one minuscule part of one of the these galaxies, reproducing molecules have evolved into the depth and richness of human conscious

experience. There is no reason to think that we at a limit of this creative process. There may be no finite limit to the evolution of physical structure and the evolution of consciousness. This is what the history of the universe, this planet and the facts of mathematics suggest to me. We need a new philosophy of mathematics grounded in our scientific understanding and the creativity that mathematics itself suggests is central to both developing mathematics and the content created in doing do.

Mathematics is both objective and creative. If a TM runs forever, this is logically determined by its program. Yet it takes creativity to develop a mathematical system to prove this. Gödel proved that no formal system that is sufficiently strong can be complete, but there is nothing (except resources) to prevent an exploration over time of every possible formalization of mathematics. As mentioned earlier, it is just such a process that created the mathematically capable human mind. The immense diversity of biological evolution was probably a necessary prerequisite for evolving that mind.

Our species has a capacity for mathematics as a genetic heritage. We will eventually exhaust what we can understand from exploiting that biological legacy through cultural evolution. This exhaustion will not occur as an event but a process that keeps making progress. However there must be a Gödel limit to the entire process even if it continues forever. Following a single path of mathematical development will lead to an infinite sequence of results all of which are encompassed in a single axiom that will never be explored. This axiom will only be explored if mathematics becomes sufficiently diverse. In the long run, the only way to avoid a Gödel limit to mathematical creativity is through ever expanding diversity.

There is a mathematics of creativity that can guide us in pursuing diversity. Loosely speaking the boundary between the mathematics of convergent processes and that of divergent creative processes is the Church-Kleene ordinal or the ordinal of the recursive ordinals. For every recursive ordinal r_0 there is a recursive ordinal $r_1 (r_0 \leq r_1)$ such that there are halting problems decidable by r_1 and not by any smaller ordinal. In turn every halting problem is decidable by some recursive ordinal. The recursive ordinals can decide the objective mathematics of convergent or finite path processes. Larger countable ordinals define a mathematics of divergent processes, like biological evolution, that follow an ever expanding number of paths[10]

The structure of biological evolution can be connected to a divergent recursive process. To illustrate this consider a TM that has an indefinite sequence of outputs that are either terminal nodes or the Gödel numbers of other recursive processes. In the latter case the TM that corresponds to the output must have its program executed and its outputs similarly interpreted. A path is a sequence of integers that corresponds to the output index at each level in the simulation hierarchy. For example the initial path segment $(4, 1, 3)$ indexes a path that corresponds to the fourth output of the root TM (r_4), the first output of r_4 ($r_{4,1}$) and the third output of $r_{4,1}$ ($r_{4,1,3}$). These paths have the structure of the tree of life that shows what species were descended from which other species.

Questions about divergent recursive processes can be of interest to inhabitants of an

[10]In a finite universe there are no truly divergent processes. Biological evolution can be truly divergent only if our universe is potentially infinite and life on earth migrates to other planets, solar systems and eventually galaxies.

always finite but potentially infinite universe. For example one might want to know if a given species will evolve an infinite chain of descendant species. In a deterministic universe. this problem can be stated using divergent recursive processes to model species. We evolved through a divergent creative process that might or might not be recursive. Quantum mechanics implies that there are random perturbations, but that may not be the final word.

Even with random perturbations, questions about all the paths a divergent recursive process can follow, may be connected to biological and human creativity. Understanding these processes may become increasingly important in the next few decades as we learn to control and direct biological evolution. Today there is intense research on using genetic engineering to cure horrible diseases. In time these techniques will become safe, reliable and predictable. The range of applications will inevitably expand. At that point it will become extremely important to have as deep an understanding as possible of what we may be doing. To learn more about this see[3].

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The defining reference for a phrase, if it exists, has the page *number* in *italics*.

This index is semiautomated with multiple entries created for some phrases and subitems automatically detected. Hand editing would improve things, but is not practical for a manual describing software and underlying theory both of which are evolving.

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Formatted: June 19, 2010